

Quantum anholonomies in time-dependent Aharonov-Bohm rings

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(Dated: August 13, 2010)

Anholonomies in eigenstates are studied through time-dependent variations of a magnetic flux in an Aharonov-Bohm ring. The anholonomies in the eigenenergy and the expectation values of eigenstates are shown to persist beyond the adiabatic regime. The choice of the gauge of the magnetic flux is shown to be crucial to clarify the relationship of these anholonomies to the eigenspace anholonomy, which is described by a non-Abelian connection in the adiabatic limit.

PACS numbers: 03.65.Vf, 03.65.Ta, 73.23.-b

I. INTRODUCTION

The parametric dependence of eigenenergies and eigenfunctions of a Hamiltonian offers a key to understanding hierarchical quantum systems, e.g., the band theory for solids and the Born-Oppenheimer approximation for molecules. Recently, it was shown that there are examples where eigenenergies and eigenfunctions are multiple-valued functions of a parameter, and cyclic adiabatic variations of the parameter transform one eigenstate into another [1]. Such phenomena are called eigenvalue and eigenspace anholonomies, or exotic quantum holonomy [2, 3]. Its applications to adiabatic manipulation of quantum states, including adiabatic quantum computation [4], were also examined recently [5, 6]. The eigenspace anholonomy was also examined through the adiabatic Floquet theory [7]. Up to now, exotic quantum holonomies were considered only in their strictly adiabatic limit. It is important to establish the robust existence of exotic quantum holonomy in the non-adiabatic realm, particularly in light of its potential experimental realization and also its potential use in quantum computation.

In this article, we present a first attempt to extend the concept of exotic quantum holonomy into time-dependent parametric motion away from the adiabatic limit. Throughout this work, we examine a charged particle in a one-dimensional ring, where a magnetic flux is applied. Although this system has been extensively investigated, as it exhibits the Aharonov-Bohm effect [8] and the “persistent current” [9], there has been no serious argument on exotic holonomies, as far as the authors are aware. It turns out that this system, with its simplicity, allows us to establish the existence of eigenvalue and eigenspace anholonomies in a time-dependent system beyond the adiabatic variation of parameters.

At the same time, we find that the freedom of the

choice of the gauge of the magnetic flux in the Aharonov-Bohm ring offers a subtle problem on the eigenspace anholonomy. In all the conventional examples, eigenvalue and eigenspace anholonomies appear in consort, which, at first sight, seems rather natural due to the correspondence of eigenvalues and eigenfunctions in Hermite operators. However, since eigenfunction itself is not an observable, there is no reason that it should follow the eigenvalue in its anholonomic variation, although any observable quantities calculated from it obviously has to show the anholonomy.

In fact, in the system we consider, the gauge transformation of the magnetic field critically changes the way the eigenstates vary under cyclic variation of the relevant parameter. As a result, the variation of the eigenfunction may not display the expected anholonomy accompanying the eigenvalue anholonomy. Indeed, it is shown that the eigenspace anholonomy appears only under a suitable choice of the vector potential of the magnetic field.

This paper is organized as follows. In Sec. II, we introduce the Aharonov-Bohm ring, and examine its properties under the gauge where the wavefunction is periodic in the ring. Due to the simplicity of this model, it suffices to examine how the eigenvalues and eigenfunctions depend on the magnetic flux, even when the magnetic flux is time-dependent. An example where the anholonomy in eigenenergies does not accompany any anholonomy in eigenfunctions is shown. In Sec. III, we examine the time evolution under the gauge devised by Byers and Yang where the gauge potential of the magnetic flux is removed from the Hamiltonian [10]. The eigenspace and the eigenenergy anholonomies are shown to be synchronized. Both adiabatic and non-adiabatic variations of the magnetic flux are examined. In Sec. IV, the parametric dependence of eigenfunctions under the Byers-Yang gauge is examined by a non-Abelian gauge connection, which is the crucial element that controls quantum anholonomies [2]. In Sec. V, we examine the adiabatic time evolution in the Byers-Yang gauge to clarify the eigenspace anholonomy in terms of a non-Abelian gauge potential, following a recent formulation for the exotic holonomy [2]. It is also shown that the dynamical phase

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in the Byers-Yang gauge has a geometric part.

II. AHARONOV-BOHM RING

We consider a quantum particle on a ring threaded by a magnetic flux of variable strength. First of all, we explain the choice of the vector potential where wavefunctions satisfy the periodic boundary condition in the ring. The particle, whose charge is q , is described by the Hamiltonian

$$H \equiv \frac{1}{2} \left[\frac{1}{i} \frac{\partial}{\partial x} - qA(x) \right]^2, \quad (1)$$

where x denotes the position of the particle on the ring ($0 \leq x < L$), and $A(x)$ is the tangent component of the vector potential at the ring. We choose the units that \hbar and the mass of the particle are unity. We assume that the deforming effect of ring curvature can be neglected. Also, we assume that the ring is so clean that the scalar potential can be ignored. Let us denote Φ be the magnetic flux, normalized by the flux quantum $2\pi/q$, applied through the ring, i.e.,

$$\Phi \equiv \frac{1}{2\pi/q} \int_0^L A(x) dx. \quad (2)$$

For the sake of simplicity, we choose $A(x) = 2\pi\Phi/qL$.

Once the gauge of the electromagnetic field is suitably chosen, this system is periodic for the normalized magnetic flux Φ with a period 1 [10], as will be shown in the next section. Accordingly, the path C where Φ is increased by its unit, say Φ' to $\Phi' + 1$, may be regarded as a closed one. It will also be shown that the anholonomies in the eigenvalues and the eigenspaces occur for the cycle C in the next section.

However, whether C is closed or open essentially depends on the choice of the gauge. In particular, C must be regarded as open under the present choice of the gauge, because the Hamiltonian [Eq. (1)] is not periodic for Φ . In the following, we show that gauge invariant quantities, such as eigenenergies and expectation values of eigenstates, depend on Φ in an aperiodic manner. This reflects the anholonomies that appear in the gauge where C is periodic.

We examine the time evolution of this system during the increment of the magnetic flux Φ by its period, i.e., from Φ' at $t = t'$ to $\Phi'' = \Phi' + 1$ at $t = t'' (> t')$.

When the system is initially in an eigenstate, the system stays in the eigenstate regardless of the speed of the variation of the magnetic field. This is because eigenfunctions of $H[\Phi(t)]$ can be chosen to be time-independent at any instant. Indeed, a normalized eigenfunction of H for a given Φ is

$$\psi_k(x) = \frac{1}{\sqrt{L}} e^{i2\pi kx/L}, \quad (3)$$

for an integer k . On the other hand, the k -th eigenenergy for a given Φ is

$$E_k(\Phi) = \frac{1}{2} \left[\frac{2\pi(k - \Phi)}{L} \right]^2. \quad (4)$$

Now it is straightforward to obtain the solution of the time-dependent Schrödinger equation

$$i \frac{\partial}{\partial t} \Psi(t, x) = H[\Phi(t)] \Psi(t, x) \quad (5)$$

with the initial condition $\Psi(t', x) = \psi_k(x)$. At the end of the path C , we obtain

$$\Psi(t'', x) = e^{i\gamma_D} \psi_k(x), \quad (6)$$

where

$$\gamma_D \equiv - \int_{t'}^{t''} E_k[\Phi(t)] dt \quad (7)$$

is the dynamical phase [11]. Hence, it is sufficient examine the Φ -dependence of eigenvalues and eigenstates to elucidate anholonomies both for adiabatic and nonadiabatic variations of Φ along C .

We examine how eigenenergies changes along C . From Eq. (4), the energy spectrum $\sigma[H(\Phi)] \equiv \{E_k(\Phi)\}_{k=-\infty}^{\infty}$ is periodic in Φ with a period 1, i.e., $\sigma[H(\Phi+1)] = \sigma[H(\Phi)]$. In this sense, C is regarded to be closed for $\sigma[H(\Phi)]$. On the other hand, each eigenvalue does not obey the periodicity, e.g.,

$$E_k(\Phi + 1) = E_{k-1}(\Phi), \quad (8)$$

which indicates the presence the eigenenergy anholonomy for the path C .

The eigenenergy anholonomy implies that an adiabatic increment of Φ along C transports the k -th eigenstate to the $k - 1$ -th eigenstate, as long as the spectrum degeneracies are not broken due to perturbations [9]. However, such an argument seems to be inconsistent with the fact that the k -th eigenfunction $\psi_k(x)$ [Eq. (3)] is independent of Φ . In fact, this does not immediately lead to any contradiction, because not only $\psi_k(x)$ but also the ray of $\psi_k(x)$ depend on the gauge of the electromagnetic field, and thus are not observables.

To characterize such an anholonomy in an eigenstate, we need to focus on the gauge invariant properties of the eigenstate. The parametric dependence of the expectation values of observables, which are gauge invariant, are consistent with the eigenenergy anholonomy [Eq. (8)]. For example, the expectation value of the velocity operator

$$v \equiv \frac{1}{i} \frac{\partial}{\partial x} - qA(x), \quad (9)$$

for the k -th eigenfunction $\psi_k(x)$ is

$$v_k(\Phi) = \frac{2\pi(k - \Phi)}{L}. \quad (10)$$

Another example is the probability current density

$$j_k(\Phi) = \frac{2\pi(k - \Phi)}{mL^2} \quad (11)$$

for the k -th eigenstate. They are consistent with the Φ -dependence of the eigenenergy $E_k(\Phi)$ [Eq. (8)], i.e.,

$$v_k(\Phi + 1) = v_{k-1}(\Phi), \quad \text{and} \quad j_k(\Phi + 1) = j_{k-1}(\Phi). \quad (12)$$

Thus it is clear that we cannot extract any Φ -dependence of the k -th eigenstate from $\psi_k(x)$ if we examine only the eigenfunction itself, and overlook the expectation values of observables. Namely, the eigenspace anholonomies of this system are described by the anholonomies in all collections of expectation values.

A remark on the gauge dependence of the Hermite operators is in order. The position operator x is gauge invariant, though its expectation value is useless in examining the anholonomy of the present system, as the expectation value for an eigenstate happens to be independent of Φ . Although the momentum operator $-i\partial_x$ is also gauge invariant, its expectation value is gauge dependent. The velocity operator (or covariant momentum operator) v [Eq. (9)] is gauge covariant and offers a key to identify the anholonomy, as shown above.

This result suggests that the standard treatment of the eigenspace anholonomy [2] is inapplicable to the present case, since the prescription in Ref. [2] essentially depends on the choice of the vector potential of the electromagnetic field. Is there any way to reconcile the present argument and the standard treatment of the eigenspace anholonomy [2]?

In the following, we shall show that an appropriate choice of the gauge of the electromagnetic field allows us to employ the prescription in Ref. [2]. A key is the use of a gauge devised by Byers and Yang [10], which provides a tool to elucidate the thermodynamic properties of Aharonov-Bohm systems. Under the Byers-Yang gauge, the parametric dependence of the eigenenergies and the eigenfunctions are associated with the eigenvalue and the eigenspace anholonomies.

III. QUASI-PERIODIC GAUGE

A proper choice of gauge transformation for the magnetic flux offers a key to resolve the question above. We show that, under a suitable gauge, the eigenvalue and eigenspace anholonomies are synchronized. Furthermore, we show that the Aharonov-Bohm ring offers an example that exhibits eigenspace anholonomy in both the adiabatic and nonadiabatic regimes.

A gauge transformation devised by Byers and Yang is defined for a wave function $\psi(x)$ that satisfies the periodic boundary condition [10]:

$$\tilde{\psi}(x) = \exp \left[-iq \int_0^x A(x') dx' \right] \psi(x). \quad (13)$$

The resultant wavefunction $\tilde{\psi}(x)$ obeys a quasi-periodic boundary condition

$$\tilde{\psi}(L) = e^{-i2\pi\Phi} \tilde{\psi}(0), \quad (14)$$

which is a manifestation of the Aharonov-Bohm effect [8]. The Hamiltonian for $\tilde{\psi}(x)$ is

$$\tilde{H}^{\text{static}} \equiv -\frac{1}{2} \frac{\partial^2}{\partial x^2}, \quad (15)$$

where we put the superscript *static* to stress that the magnetic flux is time-independent. As for the time-dependent case, we refer to Eq. (42) in Sec. V.

Here we show that this system is periodic in Φ with period 1 under the Byers-Yang gauge [10]. First, $\tilde{H}^{\text{static}}$ [Eq. (15)] is independent of Φ . Second, the quasi-periodic boundary condition [Eq. (14)] itself is periodic in Φ with period 1. Hence the periodicity of the system is evident. The path C where Φ is increased by its unit is closed under the Byers-Yang gauge.

The most crucial difference between the Byers-Yang gauge and the previous choice of the magnetic gauge (we call it the periodic gauge in the following) is seen in eigenfunctions. A k -th eigenfunction of $\tilde{H}^{\text{static}}$ in the Byers-Yang gauge is

$$\tilde{\psi}_k(x; \Phi) \equiv \frac{1}{\sqrt{L}} \exp \left[i \frac{2\pi(k - \Phi)x}{L} + i\pi\Phi \right], \quad (16)$$

where we choose the second term in the exponent so as to satisfy the parallel transport condition [12] for Φ , i.e.,

$$\langle \tilde{\psi}_k(x; \Phi), \partial_\Phi \tilde{\psi}_k(x; \Phi) \rangle = 0. \quad (17)$$

We show the eigenspace anholonomy for an adiabatic closed path C . It is sufficient to examine the parametric dependence on the eigenfunction thanks to the assumption of adiabaticity. From Eq. (16), we have

$$\tilde{\psi}_k(x; \Phi' + 1) = e^{i\pi} \tilde{\psi}_{k-1}(x; \Phi'). \quad (18)$$

This means that the adiabatic time evolution of the state vector whose initial condition is obtained as $\tilde{\psi}_{k-1}(x; \Phi')$, apart from the phase factor. This immediately indicates the presence of the eigenspace anholonomy. Namely, the state vector that initially belongs to the k -th eigenspace is adiabatically transported to the $k - 1$ -th eigenspace along a periodic increment of Φ by unity.

We extend our analysis beyond the adiabatic regime. Namely, we examine the time evolution of the wave function in the Byers-Yang gauge along the time-development of the magnetic flux $\Phi(t)$ from Φ' at $t = t'$ to Φ'' at $t = t''$. We here assume that $\Phi(t)$ gently starts and stops at $t = t'$ and t'' , respectively, to ensure the applicability of Eq. (15) at both ends. The initial wavefunction is assumed to be the k -th eigenfunction of $\tilde{H}^{\text{static}}$ at $t = t'$, i.e.,

$$\tilde{\psi}_k(x; \Phi') \quad (19)$$

In the periodic gauge, the initial wavefunction is $e^{i\pi\Phi'}\psi_k(x)$, which is an eigenfunction of Eq. (1) for an arbitrary strength of the magnetic flux. Hence, the wavefunction in the periodic gauge at $t = t''$ is $e^{i\gamma_D}e^{i\pi\Phi'}\psi_k(x)$, where γ_D is the dynamical phase, as previously shown in Eq. (7). The final wavefunction in the Byers-Yang gauge is

$$e^{i\gamma_D}e^{-i\pi(\Phi''-\Phi')}\tilde{\psi}_k(x;\Phi''), \quad (20)$$

which is an eigenstate of $\tilde{H}^{\text{static}}$ at $t = t''$. Note that the final wavefunction, except its dynamical phase γ_D , is independent of the precise time dependence of Φ .

Here we make a remark on the second factor in Eq. (20) in light of the adiabatic change of Φ . For parameters other than the electromagnetic field, the parallel transport condition ensures the correspondence of the parametric dependence of eigenstates and the adiabatic time evolution [13]. In our case, although $\tilde{\psi}_k(x;\Phi)$ [Eq. (16)] satisfies the parallel transport condition, Eq. (20) tells us that the wavefunction acquires an extra phase $-\pi(\Phi'' - \Phi')$. We shall clarify the origin of the extra phase in Sec. V, where we need to carefully examine the adiabatic time evolution in the Byers-Yang gauge.

We apply the above result to examine the eigenspace anholonomy for the closed path C . From Eq. (20), we obtain the wavefunction at $t = t''$ as

$$\tilde{\psi}_k^{(C)}(x;\Phi') \equiv e^{-i\pi}\tilde{\psi}_k(x;\Phi' + 1), \quad (21)$$

where the dynamical phase factor is excluded.

In order to compare the final wavefunction $\tilde{\psi}_k^{(C)}(x;\Phi')$ with the initial eigenfunctions, we look at the parametric dependence of $\tilde{\psi}_k(x;\Phi' + 1)$ with Φ' . From Eq. (18), we obtain the final wavefunction in terms of the initial eigenfunctions as

$$\tilde{\psi}_k^{(C)}(x;\Phi') = \tilde{\psi}_{k-1}(x;\Phi'). \quad (22)$$

We are now ready to characterize the eigenspace anholonomy by the holonomy matrix $M(C)$ whose (k'', k') -th element is the overlapping integral between $\tilde{\psi}_{k''}(x;\Phi')$ and $\tilde{\psi}_{k'}^{(C)}(x;\Phi')$ [2, 14], i.e.,

$$M_{k'',k'}(C) = \langle \tilde{\psi}_{k''}(x;\Phi'), \tilde{\psi}_{k'}^{(C)}(x;\Phi') \rangle. \quad (23)$$

From Eq. (22), we obtain

$$M_{k'',k'}(C) = \delta_{k'',k'-1}. \quad (24)$$

Hence $M(C)$ is a permutation matrix, which precisely describes the eigenspace anholonomy.

We show that all the anholonomies in the expectation values of observable can be represented by the eigenspace anholonomy. The link between the eigenspace anholonomy, and, the anholonomies in the eigenenergy and the expectation values of eigenstates are restored in the Byers-Yang gauge. We emphasize that this result is valid for arbitrary time dependence of $\Phi(t)$, i.e., both for adiabatic and nonadiabatic ones, as long as $\Phi(t)$ starts and stops gently enough at both ends.

IV. A NON-ABELIAN CONNECTION

The argument in the previous section heavily depends on the choice of the phase factor of $\tilde{\psi}_k(x;\Phi)$ in Eq. (16). We take into account such an arbitrariness by using a non-Abelian gauge connection

$$\tilde{A}_{k'',k'}(\Phi) \equiv \langle \tilde{\psi}_{k''}(x;\Phi), i\partial_\Phi \tilde{\psi}_{k'}(x;\Phi) \rangle, \quad (25)$$

which is induced by $\tilde{\psi}_k(x;\Phi)$ [2]. Note that the term “gauge” for Eq. (25) is analogous, but different from the gauge for the magnetic flux of the present system.

The non-Abelian gauge connection $\tilde{A}(\Phi)$ describes the infinitesimal change of basis vectors $\{\tilde{\psi}_k(x;\Phi)\}_k$ [15]. Namely, $\tilde{\psi}_k(x;\Phi)$ satisfies a differential equation

$$i\frac{\partial}{\partial\Phi}\tilde{\psi}_k(x;\Phi) = \sum_{k'}\tilde{\psi}_{k'}(x;\Phi)\tilde{A}_{k',k}(\Phi). \quad (26)$$

The solution of this equation for an “initial” condition $\tilde{\psi}_{k'}(x;\Phi')$, against a variation of Φ along a path C from Φ' to Φ'' , is

$$\tilde{\psi}_k(x;\Phi'') = \sum_{k'}\tilde{\psi}_{k'}(x;\Phi')W_{k',k}(C), \quad (27)$$

where

$$W(C) \equiv \exp_{\rightarrow} \left(-i \int_C \tilde{A}(\Phi) d\Phi \right), \quad (28)$$

and \exp_{\rightarrow} is the anti-path-ordered exponential.

A change of the phase factors in eigenfunctions

$$\tilde{\psi}_k(x;\Phi) \mapsto \tilde{\psi}_k(x;\Phi)e^{i\eta_k(\Phi)} \quad (29)$$

induces the following changes [2, 3]

$$\tilde{A}_{k'',k'}(\Phi) \mapsto e^{-i[\eta_{k''}(\Phi)-\eta_{k'}(\Phi)]}\tilde{A}_{k'',k'}(\Phi) - \frac{\partial\eta_k(\Phi)}{\partial\Phi}\delta_{k'',k'}, \quad (30)$$

$$W_{k'',k'}(C) \mapsto e^{-i\eta_{k''}(\Phi')}W_{k'',k'}(C)e^{i\eta_{k'}(\Phi')}. \quad (31)$$

It is also straightforward to see the covariance of the holonomy matrix (24) against the above change

$$M_{k'',k'}(C) \mapsto e^{-i\eta_{k''}(\Phi')}M_{k'',k'}(C)e^{i\eta_{k'}(\Phi')}. \quad (32)$$

An example of the evaluation of Eq. (28) is shown. Using a choice of the eigenfunction in Eq. (16), we obtain the gauge connection in the Byers-Yang gauge

$$\tilde{A}_{k'',k'}(\Phi) = \frac{i}{k''-k'}(1-\delta_{k'',k'}), \quad (33)$$

which is denoted as $\tilde{A}_{k'',k'}$, since it happens to be independent of Φ . We obtain the W -matrix [Eq. (28)] using a Fourier transformation of $\tilde{A}_{k'',k'}$:

$$\tilde{A}(\theta'',\theta') = \frac{1}{2\pi} \sum_{k''=-\infty}^{\infty} \sum_{k'=-\infty}^{\infty} e^{ik''\theta''} A_{k'',k'} e^{-ik'\theta'}. \quad (34)$$

For $0 \leq \theta', \theta'' < 2\pi$, we have

$$\tilde{A}(\theta'', \theta') = (\theta'' - \pi)\delta(\theta'' - \theta'), \quad (35)$$

from the Poisson summation formula [16],

$$\sum_{k=-\infty}^{\infty} e^{ik\theta} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\theta - 2\pi m). \quad (36)$$

Hence, we may say that \tilde{A} is a diagonal operator in “ θ -representation”. Accordingly, we have

$$[\exp(-ix\tilde{A})](\theta'', \theta') = e^{-ix(\theta'' - \pi)}\delta(\theta'' - \theta'). \quad (37)$$

Using the inverse Fourier transformation, we obtain

$$W_{k'', k'}(C) = e^{-i(k'' - k')\pi} \frac{\sin[\pi(\Phi'' - \Phi' + k'' - k')]}{\pi(\Phi'' - \Phi' + k'' - k')}. \quad (38)$$

For a periodic increment of Φ , we obtain

$$W_{k'', k'}(C) = e^{i\pi} \delta_{k''+1, k'}. \quad (39)$$

This offers a way to obtain Eq. (18) from the non-Abelian gauge connection $\tilde{A}(\Phi)$.

We explain how the prescription with a non-Abelian connection is inapplicable to the periodic gauge in Sec. II. Because the eigenfunction $\psi_k(x)$ in the periodic gauge is independent of Φ [see, Eq. (3)], the non-Abelian connection is trivial, i.e.,

$$\begin{aligned} A_{k'', k'}(\Phi) &\equiv \langle \psi_{k''}(x), i \frac{\partial}{\partial \Phi} \psi_{k'}(x) \rangle \\ &= 0. \end{aligned}$$

Although C is an open path under the periodic gauge, it is straightforward to extend the prescription to obtain the holonomy matrix from the non-Abelian connection [2, 3]. Namely, for an adiabatic change of Φ along C , the holonomy matrix is

$$\exp \left(-i \int_C A(\Phi) d\Phi \right) = 1, \quad (40)$$

where we use the fact that $A(\Phi)$ satisfies the parallel transport condition $A_{k, k}(\Phi) = 0$. Hence the holonomy matrix in the periodic gauge has nothing to do with the anholonomy, although this is consistent with the fact that $\psi_k(x)$ is independent of Φ .

V. A GEOMETRIC SIGNIFICANCE OF DYNAMICAL PHASE

So far, we have examined the time evolution of time-dependent Aharonov-Bohm ring in the Byers-Yang gauge rigorously, using the fact that eigenfunctions in the periodic gauge at each instant are independent of the magnetic flux. In this section, we show an alternative analysis of the time evolution in the Byers-Yang gauge. We

start from the time-dependent Schrödinger equation in Byers-Yang gauge, and we keep employing the Byers-Yang gauge throughout the evolution of the system. A subtle point in the separation of geometric and dynamical phases is to be elucidated. On the other hand, due to the difficulty of the problem, our analysis is restricted within the adiabatic time evolution.

We first consider the time-dependent Schrödinger equation in the Byers-Yang gauge. In the following, we explicitly denote the time-dependence of $A(x)$ as $A_t(x)$. Suppose $\tilde{\psi}(x, t)$ is obtained by the Byers-Yang gauge transformation [Eq. (13)] of a time-dependent wavefunction that satisfies the Schrödinger equation in the periodic gauge. The time-dependent Schrödinger equation for $\tilde{\psi}(x, t)$ is

$$i \frac{\partial}{\partial t} \tilde{\psi}(x, t) = \left[-\frac{1}{2} \frac{\partial^2}{\partial x^2} + q \int_0^x \frac{\partial A_t(x')}{\partial t} dx' \right] \tilde{\psi}(x, t). \quad (41)$$

Note that an extra term arises from the time-dependence of the vector potential. Hence the time evolution of $\tilde{\psi}(x, t)$ is described by the Hamiltonian

$$\tilde{H}(t) = -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{2\pi x}{L} \frac{d\Phi(t)}{dt}, \quad (42)$$

where the second term represents the effect of the time-dependence of the magnetic flux [17].

We now approximately solve the time-dependent Schrödinger equation

$$i \frac{\partial}{\partial t} \tilde{\Psi}_k(t) = \tilde{H}(t) \tilde{\Psi}_k(t) \quad (43)$$

under the quasi-periodic boundary condition (14) and the initial condition $\tilde{\Psi}_k(t') = \tilde{\psi}_k(x; \Phi')$. Here we assume that the magnetic flux adiabatically depends on time. This justifies the assumption that $\tilde{\psi}_k(x; \Phi(t))$ approximates well an eigenfunction of $\tilde{H}(t)$. Because $\psi_k(x; \Phi)$ satisfies the parallel transport condition, the final wavefunction, except its dynamical phase factor, is $\tilde{\psi}_k(x; \Phi'')$ [13]. In order to obtain a good approximation of the dynamical phase, on the other hand, we need to take into account the leading correction of the eigenenergy of $\tilde{H}(t)$

$$\tilde{E}_k(t) = E_k[\Phi(t)] + \pi \frac{d\Phi(t)}{dt}, \quad (44)$$

where

$$\tilde{E}_k(t) \equiv \langle \tilde{\psi}_k(x, \Phi(t)), \tilde{H}(t) \tilde{\psi}_k(x, \Phi(t)) \rangle. \quad (45)$$

The second term of Eq. (44) offers a nontrivial correction to the dynamical phase. Indeed, its integration

$$\int_{t'}^{t''} \pi \frac{d\Phi(t)}{dt} dt = \pi(\Phi'' - \Phi') \quad (46)$$

does not vanish even in the adiabatic limit $d\Phi/dt \rightarrow 0$. This is the origin of the second factor in Eq. (20). Although this factor is interpreted as a part of dynamical

phase factor in the Byers-Yang gauge, this is classified as a part of the geometric factor in the calculation through the periodic gauge. In any case, we obtain Eq. (20) from the adiabatic time evolution in the Byers-Yang gauge.

Hence, the adiabatic time evolution of the quantum state whose initial condition is the k -th eigenstate is

$$\tilde{\Psi}_k(t'') \equiv \exp \left[-i \int_{t'}^{t''} \tilde{E}_k(t) dt + i \int_C \tilde{A}_{kk}(\Phi) d\Phi \right] \times \tilde{\psi}_k(x, \Phi''), \quad (47)$$

where $A_{kk}(\Phi)$ is Mead-Truhlar-Berry's gauge connection for the k -th eigenfunction $\tilde{\psi}_k(x, \Phi)$ [11, 18, 19]. From Eq. (46), we have

$$\tilde{\Psi}_k(t'') = e^{i\gamma_D - i\pi(\Phi'' - \Phi') + i \int_C \tilde{A}_{kk}(\Phi) d\Phi} \tilde{\psi}_k(x, \Phi''). \quad (48)$$

Excluding the dynamical phase in the periodic gauge γ_D , we define “the geometric part” of $\tilde{\Psi}_k(t'')$ as

$$\tilde{\Psi}_k^{(g)}(t'') = e^{-i\pi(\Phi'' - \Phi') + i \int_C \tilde{A}_{kk}(\Phi) d\Phi} \tilde{\psi}_k(x, \Phi''). \quad (49)$$

The holonomy matrix for the adiabatic approximation is

$$M_{k'', k'}^{(g)} \equiv \langle \tilde{\psi}_{k''}(x, \Phi'), \tilde{\Psi}_{k'}^{(g)}(t'') \rangle. \quad (50)$$

In terms of the non-Abelian gauge connection $\tilde{A}(\Phi)$ [Eq. (25)], we obtain

$$M_{k'', k'}^{(g)}(C) = e^{-i\pi(\Phi'' - \Phi')} \left[\exp \left(-i \int_C \tilde{A}(\Phi) d\Phi \right) \right]_{k'', k'} \times \exp \left[i \int_C \tilde{A}_{k'k'}(\Phi) d\Phi \right], \quad (51)$$

where Eqs. (27) and (28) are used. In the conventional approach of the eigenspace anholonomy, the holonomy matrix is described solely by the non-Abelian gauge connection $\tilde{A}(\Phi)$ [2]. However, $M_{k'', k'}^{(g)}$ has an extra factor that comes from the dynamical phase in the Byers-Yang gauge. This factor is required to keep the consistency with the analysis in Sec. III. To see this, we evaluate $M_{k'', k'}^{(g)}$ using the choice of eigenfunctions in Eq. (16). From the evaluation of the W -matrix [Eq (38)] and the parallel transport condition $\dot{A}_{k'k'}(\Phi) = 0$ [see, Eq. (33)], it is shown that $M_{k'', k'}^{(g)}(C)$ agrees with $M_{k'', k'}(C)$ [Eq. (24)].

We finally remark that the eigenenergies of the Hamiltonian [Eq. (42)] have avoided crossings, which were ignored in the above arguments. Because the spectral degeneracies in the unperturbed Hamiltonian are lifted by the perturbation, i.e., the second term in Eq. (42), we need to take into account their effect on the adiabatic time evolution. Here, the nonadiabatic transition across an avoided crossing corresponds to the event that the system is kept to stay in an approximate eigenstate $\tilde{\psi}_k(x, \Phi)$. Because the magnitude of the perturbation is proportional to $d\Phi/dt$, the nonadiabatic transition probability is unity in the adiabatic limit. This gives a justification to our assumption of adiabatic change along $\tilde{\psi}_k(x, \Phi(t))$, and as a result, also justifies our procedural assumption to neglect the seemingly possible occurrence of avoided crossing in the adiabatic limit.

VI. SUMMARY AND DISCUSSION

We have shown that the Aharonov-Bohm ring with a vanishing electrostatic potential offers an example of anholonomies of eigenvalue and eigenstates. In particular, this system offers an extension of the quantum anholonomies for the nonadiabatic regime. At the same time, it is shown that the appearance of the eigenspace anholonomy depends on the choice of the vector potential of the magnetic flux. It is also shown that the holonomy matrix, which is the central object in the conventional prescription of the eigenspace anholonomy, offers a sensible answer only under the Byers-Yang gauge. Although this is legitimate because all the anholonomy of the present example is summarized as the eigenspace anholonomy under the Byers-Yang gauge, it is desirable to develop a prescription that is manifestly independent of the choice of the vector potential, instead of such an *ad hoc* argument. A possible strategy would be to develop the “gauge theory” for the anholonomies in eigenenergies, or in the collection of the expectation values of eigenstates. We leave this as an open question. The conventional examples of eigenvalue and the eigenspace anholonomies requires a rank-1 perturbation and its cousins [1, 2, 5, 20, 21]. In contrast to this, the eigenspace anholonomy in the present example emerges from the quasi-periodic boundary condition. Thus another subtleness of the vector potential in the quantum theory [8] is revealed.

ACKNOWLEDGMENTS

AT wishes to thank Fumihiko Nakano for a useful conversation. This work has been partially supported by the Grant-in-Aid for Scientific Research of MEXT, Japan (Grants No. 22540396 and 21540402).

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